Unit 2: Matrix

2.1 System of Linear Equations

Definition: Linear Equations

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is called a **linear equation** in n variables $x_1, x_2, \cdots, x_n$, where $a_i \in \mathbb{R}$ is the coefficient of x_i, for $i = 1, \cdots, n$ and $b \in \mathbb{R}$ is the constant term.

Example 1.1.1

1. $x = 2$ lis a inear equation in one variable x.
2. $x + y = 3$ is a linear equation in two variables x and y.
3. $x^2 + y = 3$ is not a linear equation.

Example 1.1.2

Find two numbers whose sum is 12 and whose positive difference is 3.

Definition:

A system of linear equations is called **consistent** if there exists at least one solution. It is called **inconsistent** if there is no solution.

Definitions:

1. $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ is called a **homogeneous linear equation** in n variables $x_1, x_2, \cdots, x_n$.
2. A **system of linear equations** in the variables, $x_1, x_2 \cdots, x_n$, is a finite collection of linear equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_j are real numbers. The above is a system of m equations in the n variables, $x_1, x_2 \cdots, x_n$. Written more simply in terms of summation notation, the above can be written in the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ i = 1, 2, 3, \cdots, m \qquad (1.1.1)$$

3. A system of linear equations is called a **homogeneous system** if the constant term of each equation in the system is equal to 0. A homogeneous system has the form

$$\begin{aligned} ca_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

where a_{ij} are scalars and x_i are variable

4. A list $(s_1, s_2, \cdots, s_n)$ is a **solution** of the system of linear equations if it satisfies all the linear equations in the system.

Practice 2.1

1. Solve the following systems of equations:

1. $\begin{cases} x + 2y = 13 \\ x - 4y = 1 \end{cases}$

2. $\begin{cases} 2x + 3y = -9 \\ -x - 5y = 8 \end{cases}$

3. $\begin{cases} -4x + 3y = 5 \\ 3x - 2y = -3 \end{cases}$

2. Solve the following systems of equations with more than two variables:

1. $\begin{cases} x - 4y - z = -20 \\ 3x + 2y + z = -8 \\ -2x + y - z = 15 \end{cases}$

2. $\begin{cases} -6x - 8y + z = -2 \\ -2x + 2y + 5z = 11 \\ 4x + 8y - 3z = -7 \end{cases}$

3. $\begin{cases} -2x + y + 3z + 3w = 3 \\ 2x + y - 3z + 2w = 1 \\ -2x + 4y - 3z - w = -6 \\ 2x - y - 3z + w = 5 \end{cases}$

CHALLENGING EXERCISE:

1. $\begin{cases} x + y + z + t = 0 \\ 4x + 2t = 0 \\ y - z = 0 \\ x + 2z = 1 \end{cases}$

2. Find the value of $x^2 + y^2 + z^2$, given:

$$\begin{cases} xy + yz = 35 \\ yz + xz = 32 \\ xz + xy = 27 \end{cases}$$