

Section 1.5: The Limit of a function

- Finding limits numerically and Graphically
- One-Sided Limits
- How can a limit fail to exist?
- Infinite limits, vertical asymptote

Example. Consider the function

$$f(x) = \frac{x-1}{x^2-1}$$

Q1. What is the domain of definition of the function $f(x)$?

Q2. What happens to the values of the function $f(x)$ when evaluated at values of x getting closer and closer to 1? but not equal to 1.

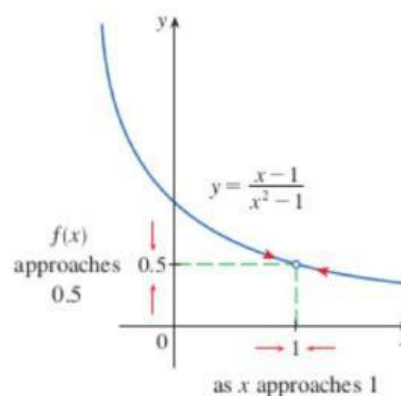
$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975


1


0.5


1


0.5



We see that from the table and the graph of $f(x)$ the closer x is to 1 (on either side of 1), the closer $f(x)$ is to 0.5.

We say that “the limit of $f(x) = \frac{x-1}{x^2-1}$ as x approaches 1 is equal to 0.5”.

Notation:

$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = 0.5$$

Suppose a function **f** is defined for all x in an open interval containing **a** (except possibly at $x = a$).

If $f(x)$ is arbitrarily close to some number L for all x sufficiently close (**but not equal**) to a , then we write

$$\lim_{x \rightarrow a} f(x) = L$$

and we say **the limit of $f(x)$ as x approaches a equals L .**

One-Sided Limits

Right-Sided limit:

Suppose the function f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \rightarrow a^+} f(x) = L$$

and we say **the limit of f as x approaches a from the right equals L .**

Left-Sided limit:

Suppose the function $f(x)$ is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \rightarrow a^-} f(x) = L$$

and we say **the limit of $f(x)$ as x approaches a from the left equals L .**

Two-Sided Limits:

The limit $\lim_{x \rightarrow a} f(x) = L$ (**two-sided limit**) exists if and only if:

$$\lim_{x \rightarrow a^+} f(x) = L \text{ and } \lim_{x \rightarrow a^-} f(x) = L$$

Remark.

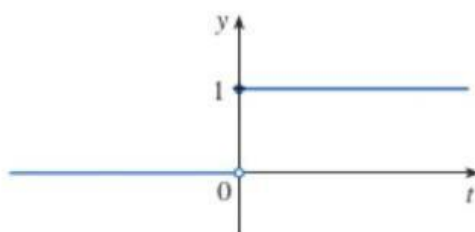
$\lim_{x \rightarrow a} f(x)$ does not exist (DNE) if and only if

- $\lim_{x \rightarrow a^+} f(x)$ or $\lim_{x \rightarrow a^-} f(x)$ does not exist, or

$$\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$$

Example.

The Heaviside function defined by: $H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$



$$\lim_{t \rightarrow 0^+} H(t) = 1, \quad \lim_{t \rightarrow 0^-} H(t) = 0,$$

The notation $t \rightarrow 0^-$ indicates that we consider only values of t that are less than 0. Likewise, $t \rightarrow 0^+$ indicates that we consider only values of t that are greater than 0.

Example. Use the graph of the function f to state the values (if they exist) of the following:

a) $\lim_{x \rightarrow 2^-} g(x) =$

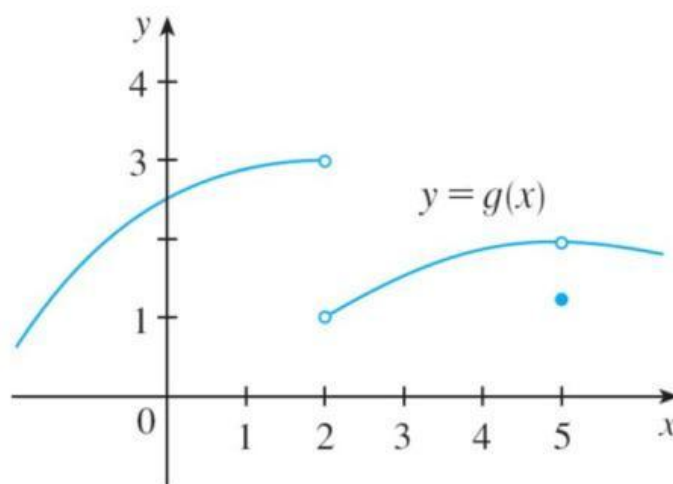
b) $\lim_{x \rightarrow 2^+} g(x) =$

c) $\lim_{x \rightarrow 2} g(x) =$

d) $\lim_{x \rightarrow 5^-} g(x) =$

e) $\lim_{x \rightarrow 5^+} g(x) =$

f) $\lim_{x \rightarrow 5} g(x) =$

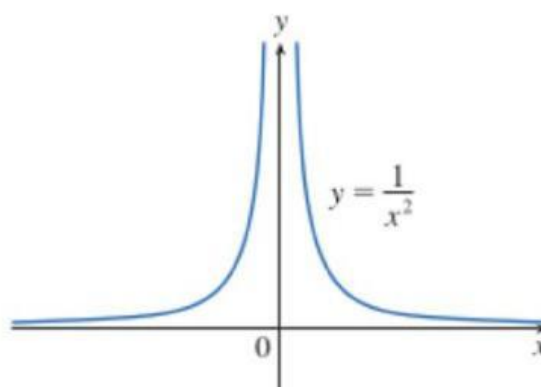


How Can a Limit Fail to Exist?

Example. Find $\lim_{x \rightarrow 0} \frac{1}{x^2}$ if it exists.

As x becomes close to 0, x^2 also becomes close to 0, and $\frac{1}{x^2}$ becomes very large. (see table below). The $\lim_{x \rightarrow 0} \frac{1}{x^2}$ does not exist (see graph)

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000



Infinite Limit ; Vertical asymptote

Definition: Let $f(x)$ be a function defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

Means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a .

Another notation for $\lim_{x \rightarrow a} f(x) = \infty$ is $f(x) \rightarrow \infty$ as $x \rightarrow a$

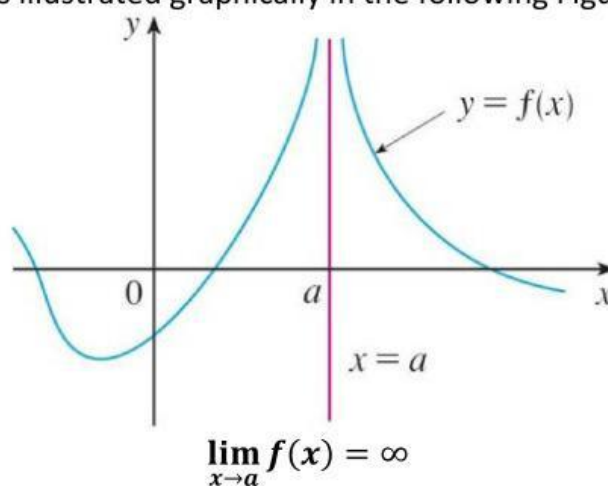
The symbol ∞ is not a number. But the expression $\lim_{x \rightarrow a} f(x) = \infty$ is often read:

“the limit of $f(x)$, as x approaches a , is infinity”

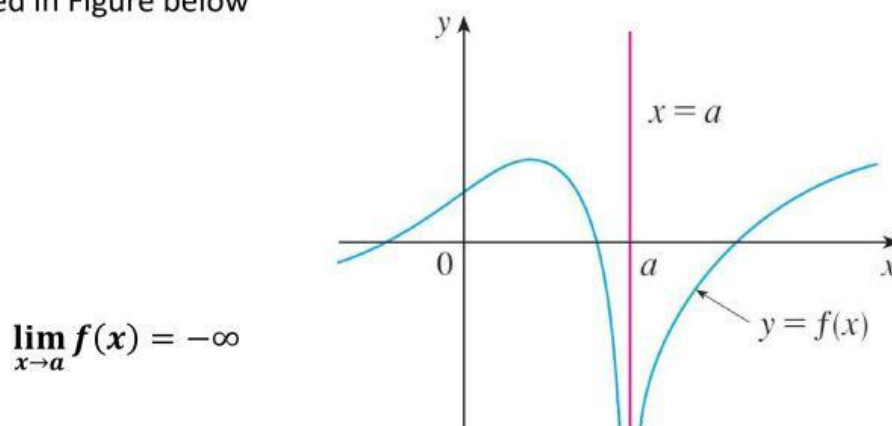
Or

“ $f(x)$ becomes infinite, as x approaches a ”

This definition is illustrated graphically in the following Figure



A similar sort of limit, for functions that become large negative as x gets close to a , illustrated in Figure below



Definition: Let $f(x)$ be a function defined on both sides of a , except possibly at a itself. Then

$$\lim_{x \rightarrow a} f(x) = -\infty$$

This means that the values of $f(x)$ can be made arbitrarily large **negative** by taking x sufficiently close to a but not equal to a .

Another notation for $\lim_{x \rightarrow a} f(x) = -\infty$ is $f(x) \rightarrow -\infty$ as $x \rightarrow a$

The symbol $-\infty$ is not a number. But the expression $\lim_{x \rightarrow a} f(x) = -\infty$ is often read:

“the limit of $f(x)$, as x approaches a , is negative infinity”

Or

“ $f(x)$ decreases without bound infinite, as x approaches a ”

Example: As an example, we have:

$$\lim_{x \rightarrow 0} \left(-\frac{1}{x^2} \right) = -\infty \quad \blacksquare$$

Remark:

Similar definitions can be given for the one-sided infinite limits

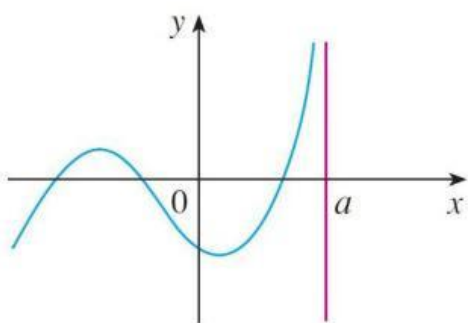
$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

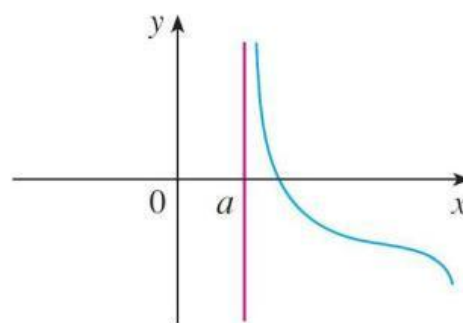
$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

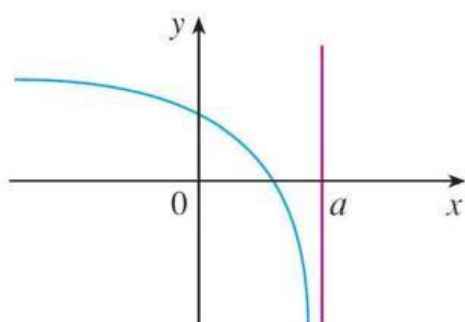
Illustrations of these four cases are given in the Figure below:



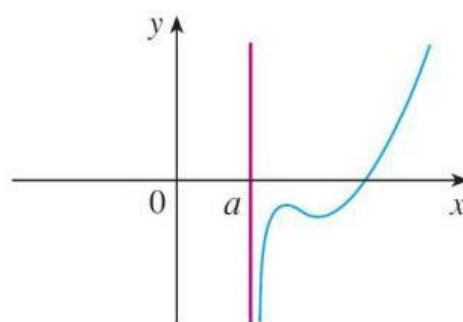
$$(a) \lim_{x \rightarrow a^-} f(x) = \infty$$



$$(b) \lim_{x \rightarrow a^+} f(x) = \infty$$



$$(c) \lim_{x \rightarrow a^-} f(x) = -\infty$$



$$(d) \lim_{x \rightarrow a^+} f(x) = -\infty$$

Definition: The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \rightarrow a} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a} f(x) = -\infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

Example: The curve $y = \frac{2x}{x-3}$ has a vertical asymptote $x = 3$

