Permutations

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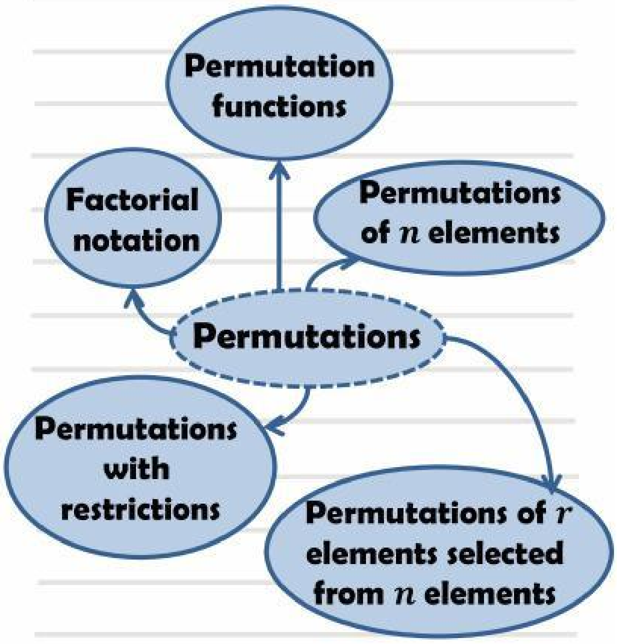
Permutations

- We can define a permutation as an ordered arrangement of some or all of the elements in a given set.
- The way set of books is arranged on a shelf, the seating positions of a group of people at a table or the way the players in a football team line up for a team photo are some examples of permutations since in each case, the order of the elements is important



A. Factorial Notation

When we are solving permutation problems, we often need to express the product of all consecutive counting numbers from 1 to a number n. Factorial notation provides an easy way to denote this product.

Definition factorial

For any counting number n, the product of all positive integers less than or equal to n is called n factorial and denoted by $n!$:

$$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$$

Example 1

$3! = 3 \cdot 2 \cdot 1 = 6$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

As a special case we accept that

$$0! = 1$$

Remark

For all positive integers, $n! = n \cdot (n-1)!$

Example 2 $7! = 7 \cdot \underbrace{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{6!} = 7 \cdot 6!$

As a result of this property, we can write

$n! = n \cdot (n-1)! = n \cdot (n-1) \cdot (n-2)!$, etc.

Example 3

Solve $\frac{(n+2)!}{(n^3-n)\cdot(n-3)!} = 6n - 12$

SOLUTION

$$\frac{(n+2)!}{(n^3-n)\cdot(n-3)!} = \frac{(n+2)\cdot(n+1)\cdot n\cdot(n-1)\cdot(n-2)\cdot(n-3)!}{n(n^2-1)\cdot(n-3)!}$$

$$= \frac{(n+2)\cdot\cancel{(n+1)}\cdot\cancel{n}\cdot\cancel{(n-1)}\cdot(n-2)\cdot\cancel{(n-3)!}}{\cancel{n}\cdot\cancel{(n-1)}\cdot\cancel{(n+1)}\cdot\cancel{(n-3)!}}$$

$$= (n+2)\cdot(n-2) = n^2 - 4$$

So $n^2-4 = 6n-12$, which gives $n^2-6n+8-4=0$.

The equation has two roots; $n_1 = 2$ and $n_2 = 4$

Since the first root makes $(n-3)$ invalid, $n = 4$

Remember!

In the factorial expression $n!$, n must be a counting number

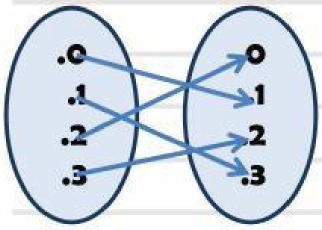
B. Permutation Functions

Definition

Let A be a non-empty set. A one-to-one and onto function from A to A is called a permutation function in A

For example, consider the function $f: A \to A$ with $A = \{0, 1, 2, 3\}$ and $f(0) = 1, f(1) = 3, f(2) = 0, f(3) = 2$

f is shown in the Venn diagram opposite.



We can see that it is a one-to-one and onto function.. and so it is a permutation function $f = \{(0,1), (1,3), (2,0), (3,2)\}$

Alternatively we can write it in the form

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \end{pmatrix}$$

This is a common way of writing a permutation function

Note that f is not the only permutation which can be defined in A in the example above. In fact, we can define $n!$ different permutation functions in a set with n elements. So in this example we can define $4! = 24$ different permutation function in A

$f_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 0 & 1 \end{pmatrix}$, $f_3 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 3 \end{pmatrix}$,

and $f_4 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

are three examples of such permutation functions.

Example 4

List all the permutation functions defined in $K = \{p, q, r\}$

SOLUTION

$$f_1 = \begin{pmatrix} p & q & r \\ p & q & r \end{pmatrix}$$

$$f_2 = \begin{pmatrix} p & q & r \\ p & r & q \end{pmatrix}$$

$$f_3 = \begin{pmatrix} p & q & r \\ q & p & r \end{pmatrix}$$

$$f_4 = \begin{pmatrix} p & q & r \\ q & r & p \end{pmatrix}$$

$$f_5 = \begin{pmatrix} p & q & r \\ r & p & q \end{pmatrix}$$

$$f_6 = \begin{pmatrix} p & q & r \\ r & q & p \end{pmatrix}$$

Permutation Functions

- Identity Permutation Functions
- Composite Permutation Functions
- The Inverse of a Permutation Function

1. Identity Permutation Functions

Definition

Let I be a permutation function defined in a set A. If $I(x) = x$ for every $x \in A$ then I is called the identity permutation function in A.

For example, if I is the identity function defined in the set $P = (1, 2, 3, 4)$ then $I(1) = 1, I(2) = 2, I(3) = 3$ and $I(4) = 4$.

We can write this identity permutation function as

$$I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

2. Composite Permutation Functions

We have already stated that a permutation function in a set A must be a one-to-one and onto function. If f and g are two permutation functions defined in A, then $f \circ g$ and $g \circ f$ are also permutation in A.

For example, suppose $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ are two permutation functions defined in the set $P = (1, 2, 3, 4)$. Then the composite function $f \circ g$ is $f \circ g(x) = f(g(x))$, so

$$f \circ g(1) = f(g(1)) = f(3) = 4$$
$$f \circ g(2) = f(g(2)) = f(2) = 1$$
$$f \circ g(3) = f(g(3)) = f(4) = 2$$
$$f \circ g(4) = f(g(4)) = f(1) = 3$$

i.e $f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

We can visualize this as

$$f \circ g = f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

EXAMPLE 5

$f = \begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix}$ and $g = \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix}$ are two permutation functions defined in $H = \{a, b, c, d\}$. Show that $f \circ g \neq g \circ f$

SOLUTION

$$f \circ g = \begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix} \circ \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}$$

$$g \circ f = \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix} \circ \begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c & d \\ d & c & b & a \end{pmatrix}$$

$$\Downarrow$$

$$f \circ g \neq g \circ f$$

Notes

1. The composition of permutation is **not commutative**: $f \circ g \neq g \circ f$
2. The composition of permutation functions is **associative**:

   $(f \circ g) \circ h = f \circ (g \circ h)$ because

   $(f \circ g) \circ h = f\big(g(h(x))\big) = f \circ (g \circ h)$
3. For any permutation f and identity permutation I in a set A.

   $f \circ I = I \circ f = f$ since

   $\forall x \in A, f \circ I(x) = f\big(I(x)\big) = f(x)$

   and $I \circ f(x) = I\big(f(x)\big) = f(x)$

$\forall x \in A$, means for all elements x in A

3. The Inverse of a pormutation Function

Since permutation f in a set A is both one-to-one and onto, by reversing the ordered pairs of f we get the inverse permutation function of f, denoted by f^{-1}

For example, if $P = \{0, 1, 2, 3\}$ is a set and $f = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix}$ is a permutation in P

Then $f^{-1} = \begin{pmatrix} 3 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$. i.e $f^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 0 \end{pmatrix}$

Note that

$$f \circ f^{-1} = f^{-1} \circ f = 1$$

EXAMPLE 6

$f = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 3 & 7 & 9 & 5 & 1 \end{pmatrix}$ **and** $g = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 3 & 7 & 9 & 5 & 1 \end{pmatrix}$

are two permutations defined in the set $K = \{1, 3, 5, 7, 9\}$**.**

$g \circ f = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 3 & 7 & 9 & 5 & 1 \end{pmatrix}$ **is given. Find** g**.**

SOLUTION

To find g**, we have to eliminate** f **from** $g \circ f$**. We can achieve this by compising** $g \circ f$ **with the inverse of** f**, since**

$$(g \circ f) \circ f^{-1} = g \circ (f \circ f^{-1}) = g \circ I = g$$

So we must find f^{-1}.

If $f = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 3 & 7 & 9 & 5 & 1 \end{pmatrix}$ **then** $f^{-1} = \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 9 & 1 & 7 & 3 & 5 \end{pmatrix}$

So $g = (g \circ f) \circ f^{-1} = \underbrace{\begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 7 & 5 & 3 & 9 & 1 \end{pmatrix}}_{given} \circ \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 9 & 1 & 7 & 3 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 3 & 5 & 7 & 9 \\ 1 & 7 & 9 & 5 & 3 \end{pmatrix}$$

C. Permutations of n Elements

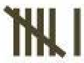
We have defined a permutation as ordered arrangement of a set of elements or items. In a permutation, the order of the items is important. We considered some permutation problems in our study of the multiplication principle.

Here is another example of a permutation problem

In how many different ways can the three students Faruq, Oleg and Evgeny be seated at a desk?

SOLUTION

Faruq Oleg Evgeny



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Definition

An ordered arrangement of some or all the elements of a given set is called a permutation

The number of permutationsof all the n distinct elements in a set is denoted by $P(n, n)$ where

$$P(n,n) = n(n-1)\cdot(n-2)\cdot \dots \cdot 1 = n!$$

In our library seating problem we can see that there are six ways for three students to sit at a desk. Using the permutation notation described above, we can write

$P(3,3) = 6$

C. Permutations of n Elements

EXAMPLE 7

What is the number of permutations of 5 different math books piled on a table?

SOLUTION

By the definition above, the answer is

$P(5,5) = 5! = 120$ **permutations**

because there are five distinct books.

We can check this answer using the counting technique we learned when we studied the multiplication principle:

1st book	2nd book	3rd book	4th book	5th book
5	4	3	2	1

Again we find that there are

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ **ways to put the nooks in a pile**

EXAMPLE 8

Murat has 5 different math books, 3 different biology books and 4 different physics books. In how many different ways can Murant arrange his books

a. on a book shelf?

b. in three different file holders, if each holder is for a different subject?

SOLUTION

a. **There is no restriction on the order of the books on the shelf so we do not need to consider the subjects. Since there are twelve books, the answer is**

$P(12,12) = 12!$ **different ways.**

b. **In this case we need to consider the subjects separatelty.**

$$\underbrace{P(5,5)}_{math} \cdot \underbrace{P(3,3)}_{biology} \cdot \underbrace{P(4,4)}_{physics} \cdot 3! = 5! \cdot 4! \cdot 3! \cdot 3!$$
$$= 120 \cdot 6 \cdot 24 \cdot 6$$
$$= 103.680$$

there are 103.680 **arrangements**

D. Permutations of r elements selected from n Elements

Many permutation problems ask us to consider arrangements of r things chosen from in things $(0 \leq r \leq n)$. i.e. Permutations of r elements chosen from a set of n elements.

EXAMPLE 9

How many different two-letter combinations can we form from the letters of the word KANO if a letter cannot be used more than once?

SOLUTION

The order of the letters is important and a letter cannot be used more than once. By the multiplication principle, the number of combinations is $4 \cdot 3 = 12$ .These combinations are

KA	AK	NK	OK
KN	AN	NA	OA
KO	AO	NO	ON

In this section we will use a new formula to solve problems on this type