10-6 Circles and Arcs

Common Core State Standards

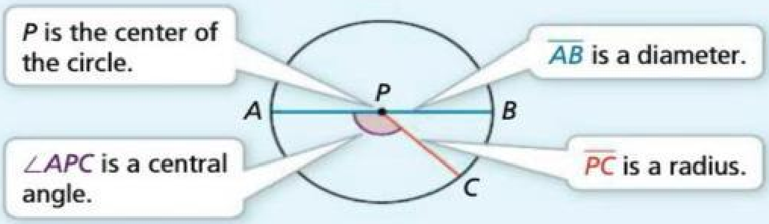
G-CO.A.1 Know precise definitions of . . . circle . . .
G-C.A.1 Prove that all circles are similar. **Also G-C.A.2, G-C.B.5**

MP 1, MP 3, MP 4, MP 6, MP 8

Objectives To find the measures of central angles and arcs
To find the circumference and arc length

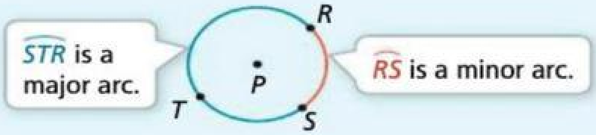
In a plane, a **circle** is the set of all points equidistant from a given point called the **center.** You name a circle by its center. Circle P ($\odot P$) is shown below.

A **diameter** is a segment that contains the center of a circle and has both endpoints on the circle. A **radius** is a segment that has one endpoint at the center and the other endpoint on the circle. **Congruent circles** have congruent radii. A **central angle** is an angle whose vertex is the center of the circle.



Essential Understanding You can find the length of part of a circle's circumference by relating it to an angle in the circle.

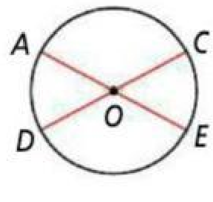
An arc is a part of a circle. One type of arc, a **semicircle,** is half of a circle. A **minor arc** is smaller than a semicircle. A **major arc** is larger than a semicircle. You name a minor arc by its endpoints and a major arc or a semicircle by its endpoints and another point on the arc.



Problem 1 Naming Arcs

A What are the minor arcs of $\odot O$?

The minor arcs are $\widehat{AD}$, $\widehat{CE}$, $\widehat{AC}$, and $\widehat{DE}$.



B What are the semicircles of $\odot O$?

The semicircles are $\widehat{ACE}$, $\widehat{CED}$, $\widehat{EDA}$, and $\widehat{DAC}$.

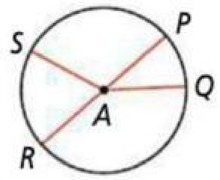
C What are the major arcs of $\odot O$ that contain point A?

The major arcs that contain point A are $\widehat{ACD}$, $\widehat{CEA}$, $\widehat{EDC}$, and $\widehat{DAE}$.

Got It? **1. a.** What are the minor arcs of $\odot A$?

b. What are the semicircles of $\odot A$?

c. What are the major arcs of $\odot A$ that contain point Q?



take note

Key Concept Arc Measure

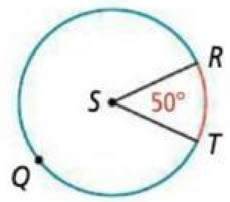
Arc Measure

The measure of a minor arc is equal to the measure of its corresponding central angle.

The measure of a major arc is the measure of the related minor arc subtracted from 360.

The measure of a semicircle is 180.

Example



$m\overset{\frown}{RT} = m\angle RST = 50$

$m\overset{\frown}{TQR} = 360 - m\overset{\frown}{RT}$

$= 310$

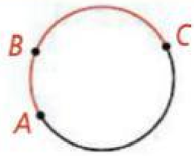
Adjacent arcs are arcs of the same circle that have exactly one point in common. You can add the measures of adjacent arcs just as you can add the measures of adjacent angles.

take note

Postulate 10-2 Arc Addition Postulate

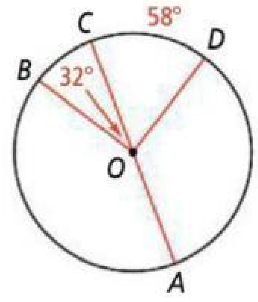
The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs.

$$m\overset{\frown}{ABC} = m\overset{\frown}{AB} + m\overset{\frown}{BC}$$



Problem 2 Finding the Measures of Arcs

What is the measure of each arc in $\odot O$?



A $\overset{\frown}{BC}$ $\quad m\overset{\frown}{BC} = m\angle BOC = 32$

B $\overset{\frown}{BD}$ $\quad m\overset{\frown}{BD} = m\overset{\frown}{BC} + m\overset{\frown}{CD}$

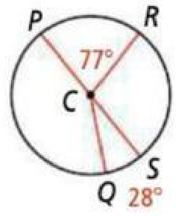
$m\overset{\frown}{BD} = 32 + 58 = 90$

C $\overset{\frown}{ABC}$ $\quad \overset{\frown}{ABC}$ is a semicircle.

$m\overset{\frown}{ABC} = 180$

D $\overset{\frown}{AB}$ $\quad m\overset{\frown}{AB} = 180 - 32 = 148$

Got It? **2.** What is the measure of each arc in $\odot C$?



a. $m\overset{\frown}{PR}$

b. $m\overset{\frown}{RS}$

c. $m\overset{\frown}{PRQ}$

d. $m\overset{\frown}{PQR}$

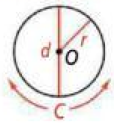
The **circumference** of a circle is the distance around the circle. The number **pi** (π) is the ratio of the circumference of a circle to its diameter.

take note

Theorem 10-9 Circumference of a Circle

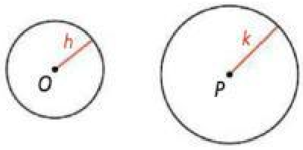
The circumference of a circle is π times the diameter.

$C = \pi d$ or $C = 2\pi r$



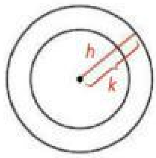
The number π is irrational, so you cannot write it as a terminating or repeating decimal. To approximate π, you can use 3.14, $\frac{22}{7}$, or the π key on your calculator.

Many properties of circles deal with ratios that stay the same no matter what size the circle is. This is because all circles are similar to each other. To see this, consider the circles at the right. There is a translation that maps circle O so that it shares the same center with circle P.



There also exists a dilation with scale factor $\frac{k}{h}$ that maps circle O to circle P. A translation followed by a dilation is a similarity transformation. Because a similarity transformation maps circle O to circle P, the two circles are similar.

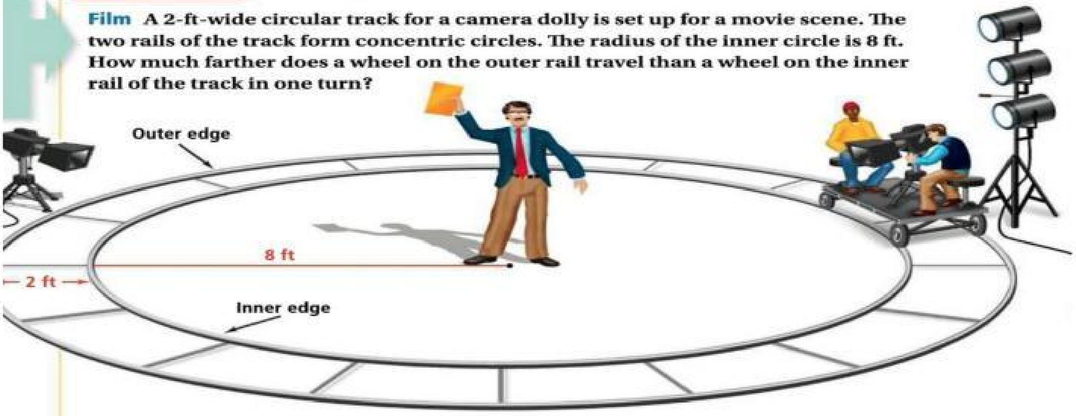
Coplanar circles that have the same center are called **concentric circles**.



Concentric circles

Problem 3 Finding a Distance

Film A 2-ft-wide circular track for a camera dolly is set up for a movie scene. The two rails of the track form concentric circles. The radius of the inner circle is 8 ft. How much farther does a wheel on the outer rail travel than a wheel on the inner rail of the track in one turn?



circumference of inner circle	$= 2\pi r$	Use the formula for the circumference of a circle.
	$= 2\pi(8)$	Substitute 8 for *r*.
	$= 16\pi$	Simplify.

The radius of the outer circle is the radius of the inner circle plus the width of the track.

radius of the outer circle $= 8 + 2 = 10$

circumference of outer circle	$= 2\pi r$	Use the formula for the circumference of a circle.
	$= 2\pi(10)$	Substitute 10 for *r*.
	$= 20\pi$	Simplify.

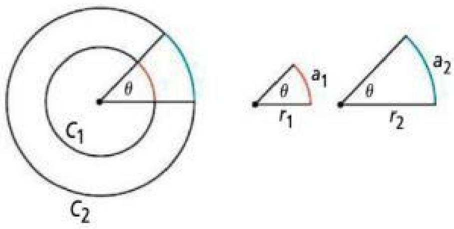
The difference in the two distances traveled is $20\pi - 16\pi$, or 4π ft.

$4\pi \approx 12.56637061$ Use a calculator.

A wheel on the outer edge of the track travels about 13 ft farther than a wheel on the inner edge of the track.

The measure of an arc is in degrees, while the **arc length** is a fraction of the circumference.

Consider the arcs shown at the right. Since the circles are concentric, there is a dilation that maps C_1 to C_2. The same dilation maps the slice of the small circle to the slice of the large circle. Since corresponding lengths of similar figures are proportional,



$$\frac{r_1}{r_2} = \frac{a_1}{a_2}$$

$$r_1 a_2 = r_2 a_1$$

$$a_1 = r_1 \frac{a_2}{r_2}$$

This means that the arc length a_1 is equal to the radius r_1 times some number. So for a given central angle, the length of the arc it intercepts depends only on the radius.

An arc of 60° represents $\frac{60}{360}$, or $\frac{1}{6}$, of the circle. So its arc length is $\frac{1}{6}$ of the circumference. This observation suggests the following theorem.

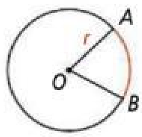
take note

Theorem 10-10 Arc Length

The length of an arc of a circle is the product of the ratio $\frac{\text{measure of the arc}}{360}$ and the circumference of the circle.

$$\text{length of } \widehat{AB} = \frac{m\widehat{AB}}{360} \cdot 2\pi r$$

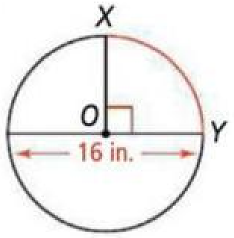
$$= \frac{m\widehat{AB}}{360} \cdot \pi d$$



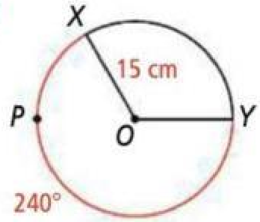
Problem 4 Finding Arc Length

What is the length of each arc shown in red? Leave your answer in terms of π.

A



B



A

length of $\widehat{XY} = \frac{m\widehat{XY}}{360} \cdot \pi d$ Use a formula for arc length.

$= \frac{90}{360} \cdot \pi(16)$ Substitute.

$= 4\pi$ in. Simplify.

B

length of $\widehat{XPY} = \frac{m\widehat{XPY}}{360} \cdot 2\pi r$

$= \frac{240}{360} \cdot 2\pi(15)$

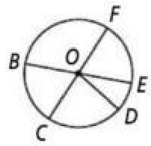
$= 20\pi$ cm

Got It? 4. What is the length of a semicircle with radius 1.3 m? Leave your answer in terms of π.

Practice and Problem-Solving Exercises MATHEMATICAL PRACTICES

Practice

Name the following in ⊙*O*. See Problem 1.

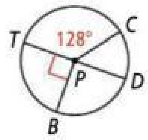


9. the minor arcs

10. the major arcs

11. the semicircles

Find the measure of each arc in ⊙*P*. See Problem 2.



12. $\overset{\frown}{TC}$ **13.** $\overset{\frown}{TBD}$ **14.** $\overset{\frown}{BTC}$

15. $\overset{\frown}{TCB}$ **16.** $\overset{\frown}{CD}$ **17.** $\overset{\frown}{CBD}$

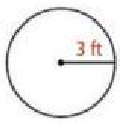
18. $\overset{\frown}{TCD}$ **19.** $\overset{\frown}{DB}$ **20.** $\overset{\frown}{TDC}$

21. $\overset{\frown}{TB}$ **22.** $\overset{\frown}{BC}$ **23.** $\overset{\frown}{BCD}$

Find the circumference of each circle. Leave your answer in terms of π. See Problem 3.

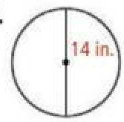
24.

25.



26.

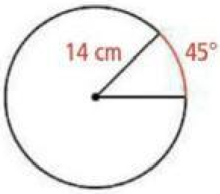
27.



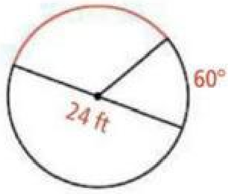
28. The camera dolly track in Problem 3 can be expanded so that the diameter of the outer circle is 70 ft. How much farther will a wheel on the outer rail travel during one turn around the track than a wheel on the inner rail?

Find the length of each arc shown in red. Leave your answer in terms of π. See Problem 4.

30.



31.



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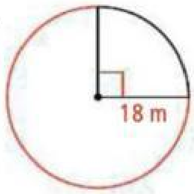

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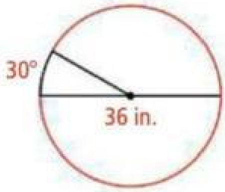

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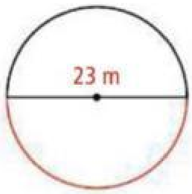

35.