Rezolvarea unor integrale prin schimbarea de variabilă

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Am selectat câteva integrale, din culegerea de probleme pentru admitere a *Universității Tehnice* din Cluj Napoca, care se rezolvă utilizând una dintre formulele schimbare de variabilă.

1. $\int_{\frac{1}{a}}^{a} \frac{\ln x}{1+x^2}dx,\quad a>0$
2. $\int_{1}^{3} \frac{\ln x}{3+x^2}dx$
3. $\int_{a}^{b} \frac{\ln x}{ab+x^2}dx,\quad a,b>0$
4. $\lim_{n\to\infty}\int_{\frac{1}{n}}^{n} \frac{arctg\, x^2}{1+x^2}dx,$
5. $\lim_{n\to\infty}\int_{0}^{n} \frac{arctgx}{x^2+x+1}dx,$
6. $\int_{\frac{1}{2}}^{2} \frac{x+\ln x}{x\left[2+\cos\left(x-\frac{1}{x}\right)\right]}dx,$

Integralele se calculează aplicând prima sau a doua formulă de variabilă, utilizând substituția $\frac{1}{x}=t$ sau $x=\frac{1}{t}$

1. $I=\int_{\frac{1}{a}}^{a}\frac{\ln x}{1+x^2}dx=\int_{a}^{\frac{1}{a}}\frac{\ln\frac{1}{t}}{1+\left(\frac{1}{t}\right)^2}\left(-\frac{1}{t^2}\right)dt=\int_{\frac{1}{a}}^{a}\frac{-\ln t}{1+t^2}dx=-I\Rightarrow 2I=0\Rightarrow I=0$

$$x=\frac{1}{t},\ dx=-\frac{1}{t^2},\ x=a\Rightarrow t=\frac{1}{a},\ x=\frac{1}{a}\Rightarrow t=a$$

2. caz particular al exercițiului 3, pentru

$$a=1,\ b=3\Rightarrow I=\frac{\pi\ln 3}{12\sqrt{3}}$$

3. Notăm

$$I=\int_a^b \frac{\ln x}{ab+x^2}dx=\int_a^b \frac{\ln x}{x^2\left(\left(\frac{\sqrt{ab}}{x}\right)^2+1\right)}dx=-\frac{1}{\sqrt{ab}}\int_a^b \frac{\ln x}{\left(\frac{\sqrt{ab}}{x}\right)^2+1}\left(-\frac{\sqrt{ab}}{x^2}\right)dx=-\frac{1}{\sqrt{ab}}\int_{\sqrt{\frac{b}{a}}}^{\sqrt{\frac{a}{b}}}\frac{\ln\frac{\sqrt{ab}}{t}}{t^2+1}dt=$$

$$u(x)=\frac{\sqrt{ab}}{x}=t,\quad u'(x)=-\frac{\sqrt{ab}}{x^2},\ u(a)=\sqrt{\frac{b}{a}},\ u(b)=\sqrt{\frac{a}{b}}$$

$$=\frac{1}{\sqrt{ab}}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}}\frac{\ln\sqrt{ab}-\ln t}{t^2+1}dt=\frac{\ln\sqrt{ab}}{\sqrt{ab}}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}}\frac{1}{t^2+1}dt-\underbrace{\frac{1}{\sqrt{ab}}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}}\frac{\ln t}{t^2+1}dt}_{0}=\frac{\ln ab}{2\sqrt{ab}}arctg\, t\Big/_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}}=\frac{\ln ab}{2\sqrt{ab}}arctg\frac{b-a}{2\sqrt{ab}}$$

4. Utilizăm egalitatea

$arctg\, x+arctg\frac{1}{x}=\frac{\pi}{2},\ \forall x>0\Rightarrow arctg\, x^2+arctg\frac{1}{x^2}=\frac{\pi}{2},\ \forall x\neq 0$, atunci :

$$I_n=\int_{\frac{1}{n}}^{n}\frac{arctg\, x^2}{1+x^2}dx=\int_{\frac{1}{n}}^{n}\frac{arctg\, x^2}{1+\frac{1}{x^2}}\frac{1}{x^2}dx=-\int_{\frac{1}{n}}^{n}\frac{arctg\, x^2}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)dx=-\int_{n}^{\frac{1}{n}}\frac{arctg\frac{1}{t^2}}{1+t^2}dt=\int_{\frac{1}{n}}^{n}\frac{arctg\frac{1}{t^2}}{1+t^2}dt=$$

$$u(x)=\frac{1}{x}=t,\ u'(x)=-\frac{1}{x^2},u\left(\frac{1}{n}\right)=n,\ u(n)=\frac{1}{n}$$

$$=\int_{\frac{1}{n}}^{n}\frac{\frac{\pi}{2}-arctg\, t^2}{1+t^2}dt=\frac{\pi}{2}\int_{\frac{1}{n}}^{n}\frac{1}{1+t^2}dt-I_n\ \Rightarrow I_n=\frac{\pi}{4}\left(arctg\, n-arctg\frac{1}{n}\right)\Rightarrow\lim_{n\to\infty}I_n=\frac{\pi^2}{8}$$

5.

$$\int_0^n\frac{arctg\, x}{x^2+x+1}dx=\int_0^{\frac{1}{n}}\frac{arctg\, x}{x^2+x+1}dx+\int_{\frac{1}{n}}^{n}\frac{arctg\, x}{x^2+x+1}dx=I_n+J_n$$

$\lim\limits_{n\to\infty}I_n=0$, deoarece $0\le x\le\frac{1}{n}<1\Rightarrow 0\le\frac{arctg\, x}{x^2+x+1}<\frac{\pi}{4},\ \forall x\in\left[0,\frac{1}{n}\right]\Rightarrow 0\le I_n<\frac{\pi}{4}\cdot\frac{1}{n}$

$$J_n=\int_{\frac{1}{n}}^{n}\frac{arctg\, x}{x^2+x+1}dx=\int_{n}^{\frac{1}{n}}\frac{arctg\frac{1}{t}}{\left(\frac{1}{t}\right)^2+\frac{1}{t}+1}\cdot\left(-\frac{1}{t^2}\right)dt=\int_{\frac{1}{n}}^{n}\frac{\frac{\pi}{2}-arctg\, t}{t^2+t+1}dt=\frac{\pi}{2}\int_{\frac{1}{n}}^{n}\frac{1}{t^2+t+1}dt-J_n\Rightarrow$$

$$J_n = \frac{\pi}{4}\int_{\frac{1}{n}}^{n} \frac{1}{t^2+t+1}dt = \frac{\pi}{4}\cdot\frac{2}{\sqrt{3}} arctg\frac{2t+1}{\sqrt{3}} /_{\frac{1}{n}}^{n} = \frac{\pi}{2\sqrt{3}}\left(arctg\frac{2n+1}{\sqrt{3}} - arctg\frac{2\cdot\frac{1}{n}+1}{\sqrt{3}} \right) \Rightarrow$$

$$\lim_{n\to\infty} J_n = \frac{\pi}{2\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \frac{\pi^2}{6\sqrt{3}} \Rightarrow \lim_{n\to\infty}\int_0^n \frac{arctgx}{x^2+x+1}dx = \frac{\pi^2}{6\sqrt{3}}$$

6. Utilizăm a doua metodă de schimbare de variabilă și obținem

$$I = \int_{\frac{1}{2}}^{2} \frac{x+\ln x}{x\left[2+\cos\left(x-\frac{1}{x}\right)\right]}dx = \int_{2}^{\frac{1}{2}} \frac{\frac{1}{t}+\ln\frac{1}{t}}{\frac{1}{t}\left[2+\cos\left(\frac{1}{t}-t\right)\right]}\left(-\frac{1}{t^2}\right)dt = \int_{\frac{1}{2}}^{2} \frac{\frac{1}{x}-\ln x}{x\left[2+\cos\left(x-\frac{1}{x}\right)\right]}dx$$

De unde $\quad 2I = \int_{\frac{1}{2}}^{2} \frac{x+\frac{1}{x}}{x\left[2+\cos\left(x-\frac{1}{x}\right)\right]}dx = \int_{\frac{1}{2}}^{2} \frac{1+\frac{1}{x^2}}{2+\cos\left(x-\frac{1}{x}\right)}dx = \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2+\cos t}dt$

Calculam integrala trigonometrică și obținem

$$I = \frac{1}{2}\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2+\cos t}dt = \frac{2}{\sqrt{3}} arctg\left(\frac{1}{\sqrt{3}} tg\frac{3}{4}\right)$$

Pentru delectare , vă propun să calculați

1. $\lim_{n\to\infty} I_n(a)$, unde $I_n(a) = \int_{\frac{1}{n}}^{n} \frac{arctg\, x}{x^2+2ax+1}dx, \quad a>0$

2. $\int_{-2}^{0} \frac{x}{\sqrt{e^x+(x+2)^2}}dx, \quad a>0$