Section 1.5: The Limit of a function

- Finding limits numerically and Graphically
- One-Sided Limits
- How can a limit fail to exist?
- Infinite limits, vertical asymptote

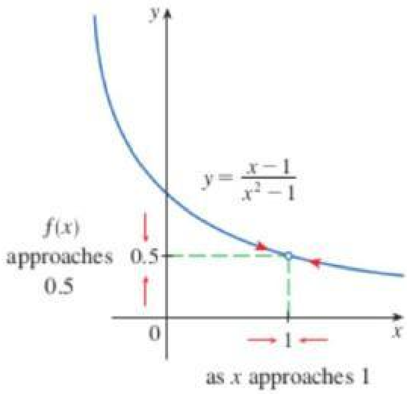
Example. Consider the function

$$f(x) = \frac{x-1}{x^2-1}$$

Q1. What is the domain of definition of the function $f(x)$?

Q2. What happens to the values of the function $f(x)$ when evaluated at values of x getting closer and closer to 1? but not equal to 1.

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975
↓ 1	↓ 0.5	↓ 1	↓ 0.5



We see that from the table and the graph of $f(x)$ the closer x is to 1 (on either side of 1), the closer $f(x)$ is to 0.5.

We say that "the limit of $f(x) = \frac{x-1}{x^2-1}$ as x approaches 1 is equal to 0.5".

Notation:

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$$

Suppose a function f **is defined for all x in an open interval containing a** (except possibly at $x = a$).

If $f(x)$ is arbitrarily close to some number L for all x sufficiently close (**but not equal**) to a, then we write

$$\lim_{x \to a} f(x) = L$$

and we say **the limit of $f(x)$ as x approaches a equals L.**

One-Sided Limits

Right-Sided limit:

Suppose the function f **is defined for all** x **near** a **with** $x > a$**.** If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

and we say **the limit of** f **as** x **approaches** a **from the right equals** L**.**

Left-Sided limit:

Suppose the function $f(x)$ **is defined for all** x **near** a **with** $x < a$**.** If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and we say **the limit of** $f(x)$ **as** x **approaches** a **from the left equals** L**.**

Two-Sided Limits:

The limit $\lim_{x \to a} f(x) = L$ **(two-sided limit)** exists if and only if:

$$\lim_{x \to a^+} f(x) = L \ \ and \ \ \lim_{x \to a^-} f(x) = L$$

Remark.

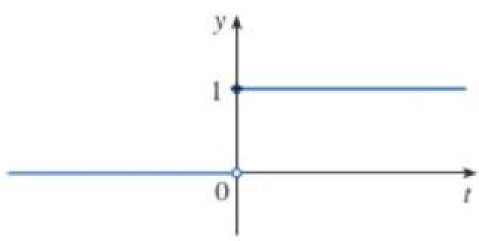
$\lim_{x \to a} f(x)$ does not exist (DNE) if and only if

- $\lim_{x \to a^+} f(x)$ **or** $\lim_{x \to a^-} f(x)$ does not exist, **or**

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

Example.

The *Heaviside* function defined by: $H(t) = \begin{cases} 0 \ \ if \ \ t < 0 \\ 1 \ \ if \ \ t \geq 0 \end{cases}$



$$\lim_{t \to 0^+} H(t) = 1, \ \ \lim_{t \to 0^-} H(t) = 0,$$

The notation $t \to 0^-$ indicates that we consider only values of t that are less than 0. Likewise, $t \to 0^+$ indicates that we consider only values of t that are greater than 0.

Example. Use the graph of the function f to state the values (if they exist) of the following:

a) $\lim_{x \to 2^-} g(x) =$

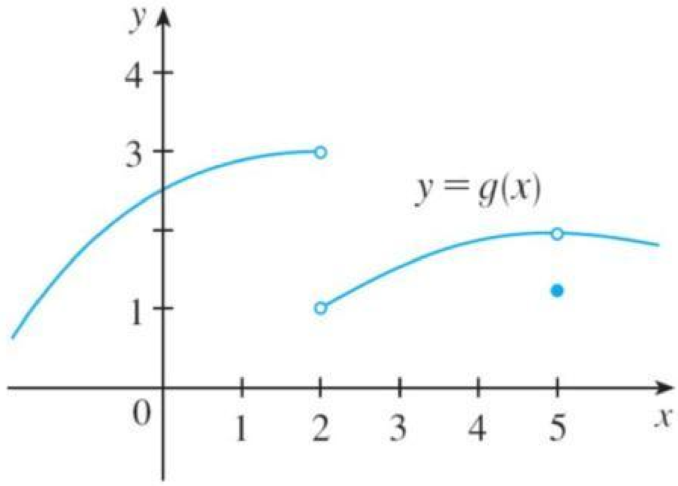
b) $\lim_{x \to 2^+} g(x) =$

c) $\lim_{x \to 2} g(x) =$

d) $\lim_{x \to 5^-} g(x) =$

e) $\lim_{x \to 5^+} g(x) =$

f) $\lim_{x \to 5} g(x) =$

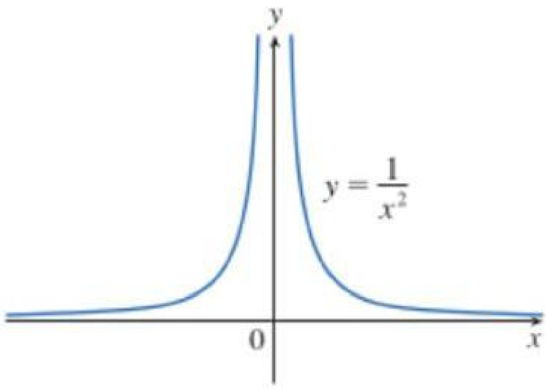


How Can a Limit Fail to Exist?

Example. Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

As x becomes close to 0, x^2 also becomes close to 0, and $\frac{1}{x^2}$ becomes very large. (see table below). The $\lim_{x \to 0} \frac{1}{x^2}$ does not exist (see graph)

x	$\frac{1}{x^2}$
±1	1
±0.5	4
±0.2	25
±0.1	100
±0.05	400
±0.01	10,000
±0.001	1,000,000



Infinite Limit ; Vertical asymptote

Definition: Let $f(x)$ be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

Means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a.

Another notation for $\lim_{x \to a} f(x) = \infty$ is $f(x) \to \infty$ as $x \to a$

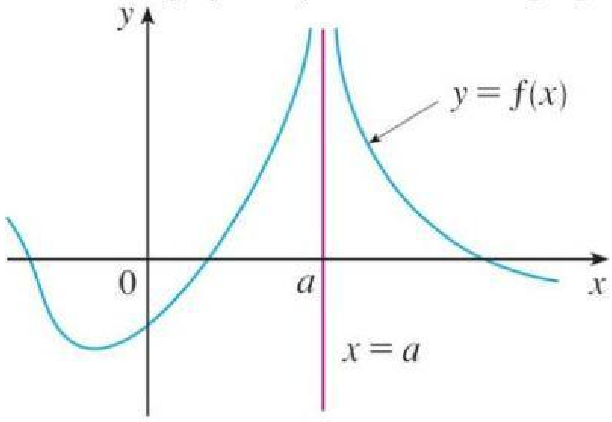
The symbol ∞ is not a number. But the expression $\lim_{x \to a} f(x) = \infty$ is often read:

"**the limit of $\boldsymbol{f(x)}$, as $\boldsymbol{x}$ approaches $\boldsymbol{a}$, is infinity**"

Or

"$\boldsymbol{f(x)}$ **becomes infinite**, as x **approaches** a "

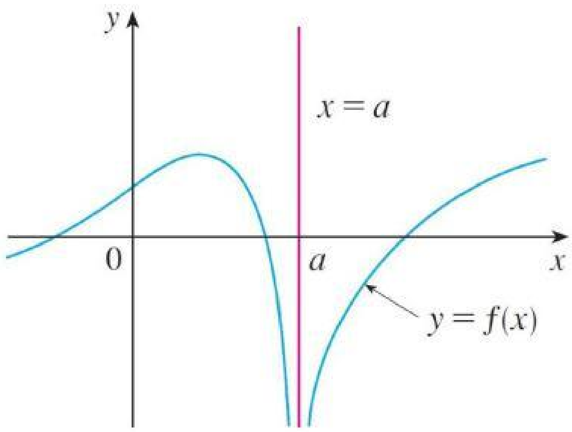
This definition is illustrated graphically in the following Figure



$$\boldsymbol{\lim_{x \to a} f(x) = \infty}$$

A similar sort of limit, for functions that become large negative as *x* gets close to *a*, illustrated in Figure below

$$\boldsymbol{\lim_{x \to a} f(x) = -\infty}$$



Definition: Let $f(x)$ be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

This means that the values of $f(x)$ can be made arbitrarily large **negative** by taking x sufficiently close to a but not equal to a.

Another notation for $\lim_{x \to a} f(x) = -\infty$ is $f(x) \to -\infty$ as $x \to a$

The symbol -∞ is not a number. But the expression $\lim_{x \to a} f(x) = -\infty$ is often read:

"**the limit of $f(x)$, as x approaches a, is negative infinity**"

Or

"**$f(x)$ decreases without bound infinite**, as x **approaches** a "

Example: As an example, we have:

$$\lim_{x \to 0} \left(-\frac{1}{x^2}\right) = -\infty \quad \blacksquare$$

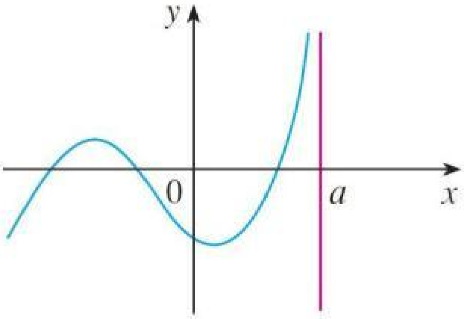
Remark:

Similar definitions can be given for the one-sided infinite limits

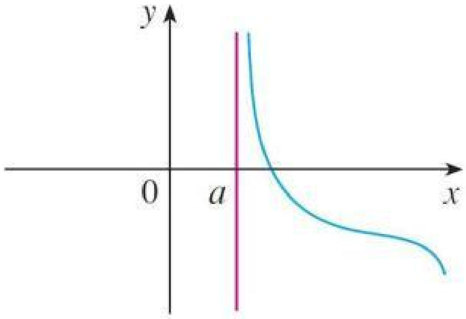
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

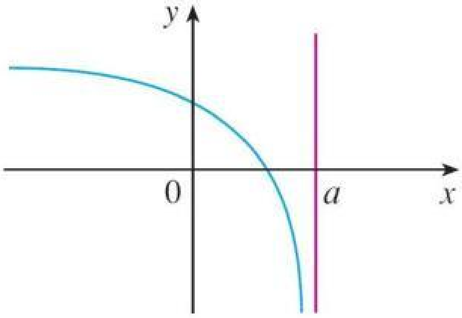
Illustrations of these four cases are given in the Figure below:



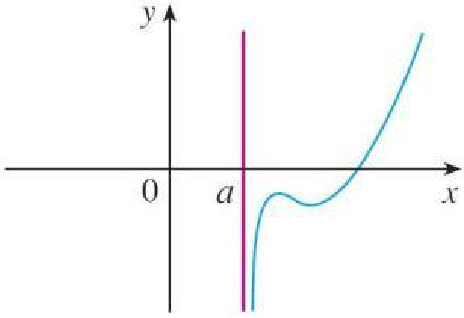
(a) $\lim_{x \to a^-} f(x) = \infty$



(b) $\lim_{x \to a^+} f(x) = \infty$



(c) $\lim_{x \to a^-} f(x) = -\infty$



(d) $\lim_{x \to a^+} f(x) = -\infty$

Definition: The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$
If at least one of the following statements is true:

$\lim_{x\to a} f(x) = \infty$ $\quad\quad$ $\lim_{x\to a^-} f(x) = \infty$ $\quad\quad$ $\lim_{x\to a^+} f(x) = \infty$

$\lim_{x\to a} f(x) = -\infty$ $\quad\quad$ $\lim_{x\to a^-} f(x) = -\infty$ $\quad\quad$ $\lim_{x\to a^+} f(x) = -\infty$

Example: The curve $y = \frac{2x}{x-3}$ has a vertical asymptote $x = 3$